41. Ako je $z = z(x,y)$ i $U(x,y) = f(g(xy) + z^2)$,

Naći $\frac{\partial^2 U}{\partial y^2}$

$$\frac{\partial U}{\partial y} = \frac{\partial}{\partial y}\left(f(g(xy) + z^2)\right) =$$

$$= f'(g(xy)+z^2) \cdot \frac{\partial}{\partial y}\left(g(xy) + z^2\right) =$$

$$= f'(g(xy)+z^2) \cdot \left(g'(xy) \cdot \frac{\partial}{\partial y}(xy) + 2z \cdot \frac{\partial z}{\partial y}\right) =$$

$$= f'(g(xy)+z^2) \cdot \left(g'(xy) \cdot x + 2z \cdot \frac{\partial z}{\partial y}\right)$$

$$\frac{\partial^2 U}{\partial y^2} = \frac{\partial}{\partial y}\left(f'(g(xy)+z^2) \cdot \left(g'(xy) \cdot x + 2z\frac{\partial z}{\partial y}\right)\right)$$

$$A = f'(g(xy) + z^2)$$

$$B = x \cdot g'(xy) + 2z\frac{\partial z}{\partial y}$$

$$\frac{\partial A}{\partial y} = f''(g(xy) + z^2) \cdot \left(g'(xy) \cdot x + 2z \cdot \frac{\partial z}{\partial y}\right)$$

$$\frac{\partial B}{\partial y} = x \cdot g''(x \cdot y) \cdot x + 2 \cdot \frac{\partial z}{\partial y} \cdot \frac{\partial z}{\partial y} + 2z \cdot \frac{\partial^2 z}{\partial y^2}$$

$$\frac{\partial^2 U}{\partial y^2} = \frac{\partial A}{\partial y} \cdot B + A \cdot \frac{\partial B}{\partial y}$$

(42) Date su glatke funkcije V i W (diferencijabilne dovoljno puta). Ako je $F(x,y,z) = V(W(V(x+y), V(z)))$ naći $\frac{\partial^2 F}{\partial z^2}$

$$F(x,y,z) = V(W(V(x+y), V(z)))$$

F (3 promj) = V (1 promj)

V → 1 promj.
W → 2 promj.

$$\frac{\partial F}{\partial z} = \overbrace{V'(W(V(x+y), V(z)))}^{A} \cdot \overbrace{\frac{\partial}{\partial z}(W(V(x+y), V(z)))}^{B}$$

neka je $s = V(x+y)$ i $t = V(z)$

$$B = \frac{\partial}{\partial z}(W(\underbrace{V(x+y)}_{s}, \underbrace{V(z)}_{t}))) =$$

$$= \frac{\partial}{\partial z}(W(s, t)) = \cancel{\frac{\partial W}{\partial s} \cdot \frac{\partial s}{\partial z}}_{0} + \frac{\partial W}{\partial t} \cdot \frac{\partial t}{\partial z}$$

$s = V(x+y) \rightarrow \frac{\partial s}{\partial z} = 0$

const tad

$$B = \frac{\partial W}{\partial s} \cdot \frac{\partial s}{\partial z} + \frac{\partial W}{\partial t} \cdot \frac{\partial t}{\partial z} = \frac{\partial W}{\partial t} \cdot V'(z)$$

$$\frac{\partial^2 F}{\partial z^2} = \frac{\partial A}{\partial z} \cdot B + A \cdot \frac{\partial B}{\partial z}$$

$$\frac{\partial A}{\partial z} = V''(W(V(x+y), V(z))) \cdot \frac{\partial W}{\partial t} \cdot V'(z)$$

$$\frac{\partial B}{\partial z} = \frac{\partial}{\partial z}\left(\frac{\partial W}{\partial t} \cdot V'(z)\right) = \frac{\partial}{\partial z}\left(\frac{\partial W}{\partial t}\right) \cdot V'(z) +$$

$$+ \frac{\partial}{\partial z}(V'(z)) \cdot \frac{\partial W}{\partial t} = \left(\frac{\partial}{\partial s}\left(\frac{\partial W}{\partial t}\right) \cdot \frac{\partial s}{\partial z} + \frac{\partial}{\partial t} \cdot \left(\frac{\partial W}{\partial t}\right) \cdot \frac{\partial t}{\partial z}\right) \cdot V'(z) +$$

0 $\quad V'(z)$

$$+ \frac{\partial W}{\partial t} \cdot V''(z)$$

$$\frac{\partial B}{\partial z} = \frac{\partial^2 W}{\partial t^2} \cdot \frac{\partial t}{\partial z} \cdot V'(z) + \frac{\partial W}{\partial t} \cdot V''(z)$$

---

43. ~~Naći diferencijal implicitno zada~~

Naći y' i y'' implicitno zadate f-je:

$$x^2 + xy + y^2 = 3$$

Neka je $F(x,y) = x^2 + xy + y^2 - 3$

Tada datu j-nu zapisujemo na sl. način: $F(x,y) = c$

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial x} = 0$$

$$\frac{\partial y}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} \longrightarrow y' = \frac{-F'_x}{F'_y}$$

$(-\infty, 0) \cup [1, +\infty)$

$F'_x = 2x + y \; ; \quad F'_y = 2y + x$

$$y' = -\frac{2x+y}{x+2y}$$

$$y'' = (y')' = \frac{(2+y')(x+2y) - (2x+y)(1+2y')}{(x+2y)^2}$$

$$y'' = -\frac{\cancel{2x} + 4y + x\cdot y' + \cancel{2yy'} - \cancel{2x} - 4xy' - y - \cancel{2yy'}}{(x+2y)^2}$$

$$y'' = -\frac{3y - 3xy'}{(x+2y)^2} = -3\,\frac{y - x\cdot y'}{(x+2y)^2} =$$

kad se uvrsti

$$= \ldots y' = -6\,\frac{x^2 + xy + y^2}{(x+2y)^3}$$

---

44. Naći diferencijal za sledeću implicitno zadatu funkciju: $\frac{x}{z} = \ln\frac{z}{y} + 1, \quad z = z(x, y)$

Neka je $F(x, y, z) = \frac{x}{z} - \ln\frac{z}{y} - 1$

Datu j-nu zapisujemo na sl. način:

$$F(x, y, z) = 0 \; \Big/ \frac{\partial}{\partial x}$$

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\cdot \cancel{\frac{\partial y}{\partial x}} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x} = 0$$

0 → jer su x i y međusobno nezavisne promjenljive

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \longrightarrow z'_x = -\frac{F'_x}{F'_z}$$

$$\frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial y} + \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} \longrightarrow z'_y = -\frac{F'_y}{F'_z}$$

$F'_x = \frac{1}{z}$

$F'_y = -\frac{y}{z} \cdot \left(-\frac{z}{y^2}\right) = \frac{1}{y}$

$F'_z = -\frac{x}{z^2} - \frac{y}{z} \cdot \frac{1}{y} = -\frac{x}{z^2} - \frac{1}{z} = \frac{-(x+z)}{z^2}$

$z'_x = \frac{\frac{1}{z}}{\frac{x+z}{z^2}} = \frac{z}{x+z}$

$z'_y = \frac{\frac{1}{y}}{\frac{x+z}{z^2}} = \frac{z^2}{y(x+z)}$

→ totalni diferencijal:

$dz = \frac{\partial z}{\partial x} \cdot dx + \frac{\partial z}{\partial y} \cdot dy$

$dz = \frac{z}{x+z} \, dx + \frac{z^2}{y(x+z)} \cdot dy$

---

45. Neka je $z = z(x,y)$ i $F\left(x + \frac{z}{y}, y + \frac{z}{x}\right) = 0$ i F diferencijabilna funkcija. Naći $\frac{\partial z}{\partial x}$ i $\frac{\partial z}{\partial y}$

→ z je fja od x i y → zadata je implicitno

* neka je $U = x + \frac{z}{y}$ a $V = y + \frac{z}{x}$

$F(U, V) = 0$

$$\boxed{z'_x = -\frac{F'_x}{F'_z} \qquad z'_y = -\frac{F'_y}{F'_z}}$$

→ ovo važi kako god da je zadata F

impl fja

$$F'_x = \frac{\partial F}{\partial U} \cdot \frac{\partial U}{\partial x} + \frac{\partial F}{\partial V} \cdot \frac{\partial V}{\partial x}$$

$$F'_x = \frac{\partial F}{\partial U} \cdot 1 + \frac{\partial F}{\partial V} \cdot \left(-\frac{z}{x^2}\right)$$

$$F'_y = \frac{\partial F}{\partial U} \cdot \frac{\partial U}{\partial y} + \frac{\partial F}{\partial V} \cdot \frac{\partial V}{\partial y}$$

$$F'_y = -\frac{z}{y^2} \cdot \frac{\partial F}{\partial U} + 1 \cdot \frac{\partial F}{\partial V}$$

$$F'_z = \frac{\partial F}{\partial U} \cdot \frac{\partial U}{\partial z} + \frac{\partial F}{\partial V} \cdot \frac{\partial V}{\partial z}$$

$$F'_z = \frac{1}{y} \cdot \frac{\partial F}{\partial U} + \frac{1}{x} \cdot \frac{\partial F}{\partial V}$$

→ imamo sve, samo uvrstimo

x, y i z tretiramo kao međusobno nezavisne promjenljive ne tretiramo $z = f(x,y)$

$F(x,y,z) = 0$

$F(x,y,z) = 2x - y^2 + 3z$

po $x \to 2$

po $y \to -2y$

po $z \to 12z^3$

---

46 Transformisati jednačinu: $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}$ uvođenjem novih promjenljivih: $U = x+y$ i $V = x-y$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial U} \cdot \frac{\partial U}{\partial x} + \frac{\partial z}{\partial V} \cdot \frac{\partial V}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial U} \cdot \frac{\partial U}{\partial y} + \frac{\partial z}{\partial V} \cdot \frac{\partial V}{\partial y}$$

$$\frac{\partial U}{\partial x} = 1; \quad \frac{\partial V}{\partial x} = 1; \quad \frac{\partial U}{\partial y} = 1; \quad \frac{\partial V}{\partial y} = -1$$

→ $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial U} + \frac{\partial z}{\partial V}$

→ $\frac{\partial z}{\partial y} = \frac{\partial z}{\partial U} - \frac{\partial z}{\partial V}$

→ uvrstimo ove izvode u j-nu

$$\frac{\partial z}{\partial U} + \frac{\partial z}{\partial V} = \frac{\partial z}{\partial U} - \frac{\partial z}{\partial V}$$

$$\frac{2\partial z}{\partial V} = 0 \to \boxed{\frac{\partial z}{\partial V} = 0}$$

ovako bi izgledala transformisana j-na

47. Transformisati jednačinu: $x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = \frac{x}{z}$

uvođenjem novih promjenljivih $U = 2x - z^2$ i $V = \frac{y}{z}$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial U} \cdot \frac{\partial U}{\partial x} + \frac{\partial z}{\partial V} \cdot \frac{\partial V}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial U} \cdot \frac{\partial U}{\partial y} + \frac{\partial z}{\partial V} \cdot \frac{\partial V}{\partial y}$$

$$\frac{\partial U}{\partial x} = 2 - 2z \cdot \frac{\partial z}{\partial x} \qquad \frac{\partial U}{\partial y} = -2z \cdot \frac{\partial z}{\partial y}$$

$$\frac{\partial V}{\partial x} = -\frac{y}{z^2} \cdot \frac{\partial z}{\partial x} \qquad \frac{\partial V}{\partial y} = \frac{1}{z} - \frac{y}{z^2} \cdot \frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial U} \cdot \left(2 - 2z \cdot \frac{\partial z}{\partial x}\right) + \frac{\partial z}{\partial V} \cdot \left(-\frac{y}{z^2} \cdot \frac{\partial z}{\partial x}\right)$$

$$\frac{2\partial z}{\partial U} = \frac{\partial z}{\partial x}\left(1 + 2z\frac{\partial z}{\partial U} + \frac{y}{z^2} \cdot \frac{\partial z}{\partial V}\right)$$

$$\boxed{\frac{\partial z}{\partial x} = \frac{2\frac{\partial z}{\partial U}}{1 + 2z \cdot \frac{\partial z}{\partial U} + \frac{y}{z^2} \cdot \frac{\partial z}{\partial U}}}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial U} \cdot \left(-2z \cdot \frac{\partial z}{\partial y}\right) + \frac{\partial z}{\partial V} \cdot \left(\frac{1}{z} - \frac{y}{z^2} \cdot \frac{\partial z}{\partial y}\right)$$

$$\left(1 + 2z\frac{\partial z}{\partial U} + \frac{y}{z^2} \cdot \frac{\partial z}{\partial V}\right) \cdot \frac{\partial z}{\partial y} = \frac{1}{z} \cdot \frac{\partial z}{\partial V}$$

$$\boxed{\frac{\partial z}{\partial y} = \frac{\frac{1}{z} \cdot \frac{\partial z}{\partial V}}{1 + 2z \cdot \frac{\partial z}{\partial U} + \frac{y}{z^2} \cdot \frac{\partial z}{\partial V}}}$$

Uvrstimo dobijene izvode u j-nu

$$\frac{2x\cdot\frac{\partial z}{\partial u}+\frac{y}{z}\cdot\frac{\partial z}{\partial v}}{1+2z\cdot\frac{\partial z}{\partial u}+\frac{y}{z^2}\cdot\frac{\partial z}{\partial v}}=\frac{x}{z}$$

$u=2x-z^2 \Rightarrow x=\frac{1}{2}(u+z^2)$

$v=\frac{y}{z}$

$$\rightarrow \frac{(u+z^2)\cdot\frac{\partial z}{\partial u}+v\cdot\frac{\partial z}{\partial v}}{1+2z\cdot\frac{\partial z}{\partial u}+\frac{v}{z}\cdot\frac{\partial z}{\partial v}}=\frac{u+z^2}{2z}$$

$$\ldots \longrightarrow \frac{\partial z}{\partial v}=\frac{z}{v}\cdot\frac{u+z^2}{z^2-u}$$

48. Naći d^2z ako je $z=\varphi(t)$ a $t=x^2+y^2$

$$dz=\frac{\partial z}{\partial x}\cdot dx+\frac{\partial z}{\partial y}\cdot dy$$

$$d^2z=\frac{\partial^2 z}{\partial x^2}\cdot dx^2+2\frac{\partial^2 z}{\partial x\partial y}dx\,dy+\frac{\partial^2 z}{\partial y^2}\cdot dy^2$$

Neka je $\varphi'=\varphi'_t$ i $\varphi''=\varphi''_t$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$\frac{\partial z}{\partial x}=\varphi'\cdot\frac{\partial t}{\partial x}=\varphi'\cdot 2x=2x\cdot\varphi'$$

$$\frac{\partial z}{\partial y}=\varphi'\cdot\frac{\partial t}{\partial y}=\varphi'\cdot 2y=2y\cdot\varphi'$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}(2x\cdot\varphi')=$$

$$=2\varphi'+2x\cdot\varphi''\cdot\frac{\partial t}{\partial x}=2\varphi'+4x^2\varphi''$$

AGATHA RUIZ DE LA PRADA

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\varphi' \cdot 2y\right) =$$

$$= 2\varphi' + 2y \cdot \varphi'' \cdot 2y = 2\varphi' + 4y^2\varphi''$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(2y\varphi'\right) = 2y\varphi'' \cdot 2x = 4xy\varphi''$$

$$d^2 z = (2\varphi' + 4x^2\varphi'')dx^2 + 2 \cdot 4xy\varphi'' dx\, dy + (2\varphi' + 4y^2\varphi'')dy^2$$

$$d^2 z = 2(dx^2 + dy^2)\varphi' + 4(x\,dx + y\,dy)^2 \cdot \varphi''$$

---

(49) Transformisati jednačinu

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$$ uvodeći nove promjenljive $U = x + y$ i $V = \frac{y}{x}$ i novu f-ju $W = \frac{z}{x}$

1. Način $$\frac{\partial W}{\partial x} = \frac{\frac{\partial z}{\partial x} \cdot x - z \cdot 1}{x^2} = \frac{1}{x} \cdot \frac{\partial z}{\partial x} - \frac{z}{x^2} \quad (1)$$

2. Način $$\frac{\partial W}{\partial x} = \frac{\partial W}{\partial U} \cdot \frac{\partial U}{\partial x} + \frac{\partial W}{\partial V} \cdot \frac{\partial V}{\partial x} = \frac{\partial W}{\partial U} - \frac{y}{x^2} \cdot \frac{\partial W}{\partial V} \quad (2)$$

Iz (1) i (2) slijedi:

$$\frac{1}{x} \cdot \frac{\partial z}{\partial x} - \frac{z}{x^2} = \frac{\partial W}{\partial U} - \frac{y}{x^2} \cdot \frac{\partial W}{\partial V}$$

$$\frac{1}{x} \cdot \frac{\partial z}{\partial x} = \frac{z}{x^2} + \frac{\partial W}{\partial U} - \frac{y}{x^2} \cdot \frac{\partial W}{\partial V} \quad / \cdot x$$

$$\frac{\partial z}{\partial x} = \frac{z}{x} + x \cdot \frac{\partial W}{\partial U} - \frac{y}{x} \cdot \frac{\partial W}{\partial V}$$

$$\frac{\partial z}{\partial x} = w + x \cdot \frac{\partial w}{\partial u} - v \cdot \frac{\partial w}{\partial v}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(w + x \cdot \frac{\partial w}{\partial u} - v \cdot \frac{\partial w}{\partial v}\right) =$$

$$= \frac{\partial w}{\partial x} + \frac{\partial w}{\partial u} + x \cdot \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial u}\right) - \frac{\partial v}{\partial x} \cdot \frac{\partial w}{\partial v} - v \cdot \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial v}\right)$$

$$= \frac{\partial w}{\partial u} - \frac{y}{x^2} \cdot \frac{\partial w}{\partial v} + \frac{\partial w}{\partial u} + x \cdot \left(\frac{\partial^2 w}{\partial u^2} \cdot \frac{\partial u}{\partial x} + \frac{\partial^2 w}{\partial v \partial u} \cdot \frac{\partial v}{\partial x}\right)$$

$$- \underbrace{\left(-\frac{y}{x^2}\right)}_{\frac{\partial v}{\partial x}} \cdot \frac{\partial w}{\partial v} + (-v) \cdot \left(\frac{\partial^2 w}{\partial u \partial v} \cdot \frac{\partial u}{\partial x} + \frac{\partial^2 w}{\partial v^2} \cdot \frac{\partial v}{\partial x}\right)$$

w fja od u i v

$\frac{\partial w}{\partial u}$ opet fja od u i v

→ kao izvod: $\frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x}$

$$= \frac{\partial w}{\partial u} - \frac{y}{x^2} \cdot \frac{\partial w}{\partial v} + \frac{\partial w}{\partial u} + x\left(\frac{\partial^2 w}{\partial u^2} \cdot \frac{\partial u}{\partial x} + \frac{\partial^2 w}{\partial v \partial u} \cdot \frac{\partial v}{\partial x}\right) +$$

$$+ \frac{y}{x^2} \cdot \frac{\partial w}{\partial v} - v \cdot \left(\frac{\partial^2 w}{\partial u \partial v} \cdot \frac{\partial u}{\partial x} + \frac{\partial^2 w}{\partial v^2} \cdot \frac{\partial v}{\partial x}\right) =$$

$$\boxed{\frac{\partial^2 z}{\partial x^2}} = 2\frac{\partial w}{\partial u} - \cancel{\frac{y}{x^2} \frac{\partial w}{\partial v}} + x \cdot \frac{\partial^2 w}{\partial u^2} - \frac{y}{x} \frac{\partial^2 w}{\partial v \partial u} +$$

$$+ \cancel{\frac{y}{x^2} \frac{\partial w}{\partial v}} - v \cdot \frac{\partial^2 w}{\partial u \partial v} + v \cdot \frac{y}{x^2} \cdot \frac{\partial^2 w}{\partial v^2} =$$

$$= 2\frac{\partial w}{\partial u} + x \cdot \frac{\partial^2 w}{\partial u^2} - 2v \cdot \frac{\partial^2 w}{\partial u \partial v} + v^2 \cdot \frac{1}{x} \cdot \frac{\partial^2 w}{\partial v^2}$$

$$W = \frac{z}{x} \qquad U = x+y; \quad V = \frac{y}{x}$$

$$\frac{\partial W}{\partial y} = \frac{1}{x} \cdot \frac{\partial z}{\partial y} \quad (3)$$

$$\frac{\partial W}{\partial y} = \frac{\partial W}{\partial U} \cdot \frac{\partial U}{\partial y} + \frac{\partial W}{\partial V} \cdot \frac{\partial V}{\partial y} = \frac{\partial W}{\partial U} + \frac{\partial W}{\partial V} \cdot \frac{1}{x} \quad (4)$$

$$\frac{\partial z}{\partial y} \cdot \frac{1}{x} = \frac{\partial W}{\partial U} + \frac{\partial W}{\partial V} \cdot \frac{1}{x} \quad / \cdot x$$

$$\frac{\partial z}{\partial y} = x \frac{\partial W}{\partial U} + \frac{\partial W}{\partial V}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(x \cdot \frac{\partial W}{\partial U} + \frac{\partial W}{\partial V}\right) =$$

$$= x \cdot \frac{\partial}{\partial y}\left(\frac{\partial W}{\partial U}\right) + \frac{\partial}{\partial y}\left(\frac{\partial W}{\partial V}\right) =$$

$$= x \left(\frac{\partial^2 W}{\partial U^2} \cdot \frac{\partial U}{\partial y} + \frac{\partial^2 W}{\partial U \cdot \partial V} \cdot \frac{\partial V}{\partial y}\right) + \left(\frac{\partial^2 W}{\partial V^2} \cdot \frac{\partial V}{\partial y} + \frac{\partial^2 W}{\partial U \partial V} \cdot \frac{\partial U}{\partial y}\right) =$$

$$\Rightarrow \frac{\partial U}{\partial y} = 1; \quad \frac{\partial V}{\partial y} = \frac{1}{x}$$

$$\rightarrow \frac{\partial^2 z}{\partial y^2} = x \cdot \left(\frac{\partial^2 W}{\partial U^2} + \frac{\partial^2 W}{\partial U \partial V} \cdot \frac{1}{x}\right) + \left(\frac{\partial^2 W}{\partial U \partial W} + \frac{\partial^2 W}{\partial V^2} \cdot \frac{1}{x}\right) =$$

$$\boxed{= x \frac{\partial^2 W}{\partial U^2} + 2 \frac{\partial^2 W}{\partial U \partial V} + \frac{\partial^2 W}{\partial V^2} \cdot \frac{1}{x} = \frac{\partial^2 z}{\partial y^2}}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(x \cdot \frac{\partial W}{\partial U} + \frac{\partial W}{\partial V}\right) =$$

$$= \frac{\partial}{\partial x}\left(x \cdot \frac{\partial W}{\partial U}\right) + \frac{\partial}{\partial x}\left(\frac{\partial W}{\partial V}\right) =$$

$$= \frac{\partial W}{\partial U} + x \cdot \left(\frac{\partial^2 W}{\partial U^2} \cdot \overset{1}{\frac{\partial U}{\partial x}} + \frac{\partial^2 W}{\partial V \partial U} \cdot \overset{-\frac{y}{x^2}}{\frac{\partial V}{\partial x}} \right) +$$

$$+ \frac{\partial^2 W}{\partial U \cdot \partial V} \cdot \frac{\partial U}{\partial x} + \frac{\partial^2 W}{\partial V^2} \cdot \frac{\partial V}{\partial x} =$$

$$= \frac{\partial W}{\partial U} + x \cdot \frac{\partial^2 W}{\partial U^2} - \cancel{x} \cdot \frac{y}{x^{\cancel{2}}} \cdot \frac{\partial^2 W}{\partial V \partial U} + \frac{\partial^2 W}{\partial U \partial V} +$$

$$- \frac{\partial^2 W}{\partial V^2} \cdot \frac{y}{x^2} = \qquad \frac{y}{x} = V$$

$$= \boxed{\frac{\partial W}{\partial U} + x \cdot \frac{\partial^2 W}{\partial U^2} - V \cdot \frac{\partial^2 W}{\partial U \partial V} + \frac{\partial^2 W}{\partial U \partial W} - \frac{V}{x} \cdot \frac{\partial^2 W}{\partial V^2}}$$

→ uvrstimo izvode u jednačinu i dobijamo:

$$\frac{V^2}{x} \cdot \frac{\partial^2 W}{\partial V^2} + \frac{2V}{x} \cdot \frac{\partial^2 W}{\partial V^2} + \frac{1}{x} \cdot \frac{\partial^2 W}{\partial V^2} = 0$$

$$U = x + y \; ; \quad V = \frac{y}{x} \rightarrow y = Vx$$

$$U = x + Vx \rightarrow x = \frac{U}{1+V}$$

$$\boxed{\frac{(V+1)^3}{U} \cdot \frac{\partial^2 W}{\partial V^2} = 0}$$

(50.) Naći izvod funkcije:

$z = 2x^2 - 3y^2$ u tački $P(1,0)$ u pravcu vektora koji sa osom Ox gradi ugao $120°$

$$\frac{\partial z}{\partial l}(P) = \frac{\partial z}{\partial x}(P) \cdot \cos\alpha + \frac{\partial z}{\partial y}(P) \cdot \cos\beta$$

α i β su uglovi koje vektor $\vec{e}$ zaklapa sa koord. osama

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\vec{l_0} = (\cos\alpha, \cos\beta) \longrightarrow \cos\beta = \sin\alpha$$

$$\alpha = 120° ; \quad \cos 120° = \frac{\sqrt{3}}{2} \qquad \sin 120° = -\frac{1}{2}$$

$$\frac{\partial z}{\partial x} = 4x \qquad \frac{\partial z}{\partial x}(P) = 4$$

$$\frac{\partial z}{\partial y} = -6y \qquad \frac{\partial z}{\partial y}(P) = 0$$

$$\frac{\partial z}{\partial l}(P) = 4 \cdot \left(-\frac{1}{2}\right) + 0 \cdot \frac{\sqrt{3}}{2} = -2$$

---

(51) Naći izvod funkcije $f(x,y) = 1 - \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)$ u tački $M\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$ u pravcu normale u toj tački na krivu $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

→ Normala na $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ u tački M → odnosno trebaju nam uglovi koje ona gradi sa koordinatnim osama

$$\frac{\partial f}{\partial l}(M) = \frac{\partial f}{\partial x}(M) \cdot \cos\alpha + \frac{\partial f}{\partial y}(M) \cdot \cos\beta$$

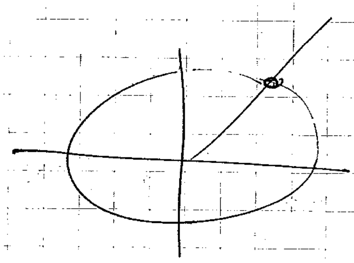
Koeficijent pravca normale je

$$k_n = \frac{-1}{y'\left(\frac{a}{\sqrt{2}}\right)}$$

→ izvod implicitno zadate fje ili y izrazimo

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2) \quad y = \pm \frac{b}{a} \cdot \sqrt{a^2 - x^2}$$



→ u 1. kvadrantu ⇒

$$\Rightarrow y = \frac{b}{a}\sqrt{a^2 - x^2}$$

$$y' = \frac{b}{a} \cdot \frac{-x}{\sqrt{a^2 - \frac{a^2}{2}}}$$ (dvojka se pokratila $\frac{-\cancel{2}x}{\cancel{2}\sqrt{\ldots}}$)

$$y'\left(\frac{a}{\sqrt{2}}\right) = \frac{b}{a} \cdot \frac{-\frac{a}{\sqrt{2}}}{\sqrt{a^2 - \frac{a^2}{2}}} = -\frac{b}{a}$$

$$k_n = -\frac{1}{-\frac{b}{a}} = \boxed{\frac{a}{b} = k_n}$$

$\alpha \to$ ugao koji normala zaklapa sa x osom;

$$tg\,\alpha = k_n = \frac{a}{b}$$

$$\cos\alpha = \frac{1}{\sqrt{1 + \frac{a^2}{b^2}}} = \frac{b}{\sqrt{a^2+b^2}}$$

$\boxed{+ \text{ uvrstiti}}$

$$\cos\beta = \sin\alpha = \frac{\frac{a}{b}}{\sqrt{1 + \frac{a^2}{b^2}}} = \frac{\frac{a}{\cancel{b}}}{\frac{\sqrt{a^2+b^2}}{\cancel{b}}} = \frac{a}{\sqrt{a^2+b^2}}$$

$$\frac{\partial f}{\partial \ell}(M) = \frac{\partial f}{\partial x}(M) \cdot \cos\alpha + \frac{\partial f}{\partial y}(M) \cdot \cos\beta$$

$$\frac{\partial f}{\partial x} = -\frac{2x}{a^2} \to \frac{\partial f}{\partial x}(M) = \frac{-2}{a^2} \cdot \frac{a}{\sqrt{2}} = -\frac{\sqrt{2}}{a} \qquad \frac{\partial f}{\partial y}(M) = -\frac{\sqrt{2}}{b}$$

AGATHA RUIZ DE LA PRADA

52) Naći gradijent funkcije $z = x^2 \cdot y$ u tački
a) → $P(1,1)$

b) Naći izvod f-je u tački $P(1,1)$ u pravcu dobijenog gradijenta

a) $\mathrm{grad}\, z\,(P) = \frac{\partial}{\partial x}(P) \cdot \vec{i} + \frac{\partial}{\partial y}(P) \cdot \vec{j}$

$\frac{\partial z}{\partial x} = 2xy \qquad \frac{\partial z}{\partial y} = x^2$

$\mathrm{grad}\, z\,(P) = 2\vec{i} + \vec{j}$

b) $\frac{\partial z}{\partial e} = |\mathrm{grad}\, \cancel{z}(P)| = \sqrt{4+1} = \sqrt{5}$

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53) Dokazati da izraz:

$(3x^2 + 3y - 1)\,dx + (z^2 + 3x)\,dy + (2yz + 1)\,dz$

predstavlja diferencijal neke funkcije f, a zatim odrediti sve moguće f

1. $P(x,y)\,dx + Q(x,y)\,dy$ je totalni diferencijal neke funkcije akko je parcijalni izvod

$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$

2. $P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$ je totalni diferencijal neke f-je akko je

$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}; \quad \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}; \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}$

neka je $P(x,y,z) = 3x^2 + 3y - 1$

$Q(x,y,z) = z^2 + 3x$

$R(x,y,z) = 2yz + 1$

$f = ?$

$$df = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y} \cdot dy + \frac{\partial f}{\partial z} \cdot dz$$

$$\frac{\partial f}{\partial x} = P \; ; \quad \frac{\partial f}{\partial y} = Q \; ; \quad \frac{\partial f}{\partial z} = R$$

$$\frac{\partial f}{\partial x} = 3x^2 + 3y - 1 \rightarrow f(x,y,z) = \int (3x^2 + 3y - 1)\, dx =$$

$$= \frac{3x^3}{3} + 3yx - x + C_1 \rightarrow C_1 \text{ zavisi od } y, z$$

diferenciramo ovo po y

$$\frac{\partial f}{\partial y} = 3x + \frac{\partial}{\partial y} C_1(y,z)$$

$$\frac{\partial f}{\partial y} = Q(x,y,z) \; \wedge \; \frac{\partial f}{\partial y} = 3x + \frac{\partial}{\partial y} C_1(y,z)$$

$$\cancel{3x} + \frac{\partial}{\partial y} C_1(y,z) = z^2 + \cancel{3x}$$

$$\frac{\partial}{\partial y} C_1(y,z) = z^2$$

$$C_1(y,z) = \int z^2 dy = yz^2 + C_2(z)$$

ne zavisi od x jer C_1 ne zavisi od x

$$f(x,y,z) = x^3 + 3xy - x + yz^2 + C_2(z)$$

$$\frac{\partial f}{\partial z} = R \; ; \quad \frac{\partial f}{\partial z} = 2yz + C_2'(z)$$

$$\cancel{2yz} + C_2'(z) = R = \cancel{2yz} + 1$$

$$C_2'(z) = 1 \longrightarrow C_2 = \int 1 \cdot dz = z + C$$

$$f(x,y,z) = x^3 + 3xy - x + yz^2 + z + C$$

AGATHA RUIZ DE LA PRADA